$U(1)$ Connection, Nonlinear Dirac-like Equations and Seiberg-Witten Equations

Liangzhong Hu[1] **and Liangyou Hu**[2]

Abstract

By analysing the work of Campolattaro we argue that the second Seiberg-Witten equation over the Spin_4^c manifold, i.e., $F_{ij}^+ = \langle M, S_{ij} M \rangle$, is the generalization of the Campolattaro's description of the electromagnetic field tensor $F^{\mu\nu}$ in the bilinear form $F^{\mu\nu} = \overline{\Psi} S^{\mu\nu} \Psi$. It turns out that the Seiberg-Witten equations (also the perturbed Seiberg-Witten equations) can be well understood from this point of view. We suggest that the second Seiberg-Witten equation can be replaced by a nonlinear Dirac-like Equation. We also derive the spinor representation of the connection on the associated unitary line bundle over the Spin_4^c manifold.

KEY WORDS: connection, curvature, spinor.

[1]Institute of Mathematics, Peking University, Beijing 100871, People's Republic of China. E-mail address: lhu@sxx0.math.pku.edu.cn

[2]Basic Department, Dongbei University of Finance and Economics, Dalian 116025, People's Republic of China.

1 INTRODUCTION

The Maxwell equations and the Dirac equation are among the most celebrated equations of physics. Campolattaro (1980a, 1980b) started with the analysis of the Maxwell equations by writing the electromagnetic field tensor $F_{\mu\nu}$ in the equivalent bilinear form

$$F^{\mu\nu} = \overline{\Psi} S^{\mu\nu} \Psi \tag{1}$$

where $\mu, \nu = 0, 1, 2, 3$. Ψ is a Dirac spinor, $\overline{\Psi} = \Psi^\dagger \gamma^0$ is the Dirac conjugation of Ψ. $S^{\mu\nu}$ is the spin operater defined by

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu] \tag{2}$$

and the γ's are the Dirac matrices satisfying

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \tag{3}$$

with $\eta^{\mu\nu}$ the Minkowski metric tensor given by $\eta^{\mu\nu} = diag(1, -1, -1, -1)$.

In this representation the dual tensor

$$*F^{\mu\nu} = \overline{\Psi} \gamma^5 S^{\mu\nu} \Psi \tag{4}$$

From now on, the Einstein sum convention is adopted throughout. The Maxwell equations read (a comma followed by an index represents the partial derivative with respect to the variable with that index)

$$(\overline{\Psi} S^{\mu\nu} \Psi)_{,\mu} = j^\nu \tag{5}$$

$$(\overline{\Psi} \gamma^5 S^{\mu\nu} \Psi)_{,\mu} = 0 \tag{6}$$

Moreover, the duality (Rainich, 1925; Misner and Wheeler, 1957) by the complexion α, namely

$$\overline{F}^{\mu\nu} = F^{\mu\nu} \cos\alpha + *F^{\mu\nu} \sin\alpha \tag{7}$$

is equivalent to a Touschek-Nishijima (Touschek, 1957; Nishijima, 1957) transformation for the spinor Ψ to the spinor Ψ' given by

$$\Psi' = e^{\gamma^5 \alpha/2} \Psi \tag{8}$$

with

$$e^{\gamma^5 \alpha} = \cos\alpha + \gamma^5 \sin\alpha \tag{9}$$

and

$$\cos\alpha = \frac{\overline{\Psi}\Psi}{\rho} \tag{10}$$

$$\sin\alpha = \frac{\overline{\Psi}\gamma^5\Psi}{\rho} \tag{11}$$

ρ being the positive square root of

$$\rho^2 = (\overline{\Psi}\Psi)^2 + (\overline{\Psi}\gamma^5\Psi)^2 \tag{12}$$

Campolattaro showed that the two spinor Maxwell equations (5) and (6) is equivalent to a single nonlinear first-order equation for the spinor, namely

$$\gamma^{\mu}\Psi_{,\mu} = -i\gamma^{\mu}\frac{e^{\gamma^5\alpha}}{\rho}\{Im(\overline{\Psi}_{,\mu}\Psi) - j_{\mu} - \gamma^5 Im(\overline{\Psi}_{,\mu}\gamma^5\Psi)\}\Psi \tag{13}$$

The relation between Dirac and Maxwell equations was also considered by Vaz and Rodrigues (1993).

On the other hand, Witten (1994) introduced the Seiberg-Witten equations. For more details, see, e. g., (Moore, 1996; Morgan, 1996). By counting the solutions of the equations with an Abelian gauge group, new invariants of 4-manifolds can be obtained. These invariants are closely related to Donaldson's polynomial invariants, but in many respects much simpler to work with. One of the Seiberg-Witten equation is the Dirac equation on the Spin_4^c manifold X,

$$\widetilde{D}M = \gamma_i\widetilde{\nabla}_i M = 0 \tag{14}$$

where the Riemannian indices $i = 1, 2, 3, 4$. $\widetilde{D}$ is the Dirac operator on the Spin_4^c manifold X. $M \in S^+$, S^+ is the positive chirality spinor bundle over X. $\widetilde{\nabla}_i$ is a covariant derivative acting on S^+. γ_i, the Clifford matrices satisfy

$$\gamma_i\gamma_j + \gamma_j\gamma_i = -2\delta_{ij} \tag{15}$$

Locally,

$$S^+ = S_0^+ \otimes L^{\frac{1}{2}} \tag{16}$$

$$\widetilde{\nabla}_i = \nabla_i \otimes 1 + 1 \otimes \nabla_i' \tag{17}$$

Here S_0^+ is the local positive chirality spinor bundle over X, $L^{\frac{1}{2}}$ is the square root of the line bundle L over X, $\nabla_i = \partial_i + \Gamma_i$, Γ_i is the induced Levi-Civita connection on S_0^+. $\nabla_i' = \partial_i + ia_i$, a_i is the connection on $L^{\frac{1}{2}}$ over X. (For the later use, notice that when the covariant derivative $\nabla_i = \partial_i + \Gamma_i$ acts on the γ's or tensors on the Riemannian manifold X, Γ_i is the Levi-Civita connection.)

The second Seiberg-Witten equation is

$$F_{ij}^+ = \langle M, S_{ij}M\rangle \tag{18}$$

Here $\langle , \rangle$ represents Hermite inner product, it is the pointwise inner product. $S_{ij} = \frac{i}{4}[\gamma_i, \gamma_j]$ is the spin operater on X, and F_{ij}^+ is the self dual part of the curvature on the line bundle over X. However, Witten did not tell us where the second equation comes from. Obviously, a profound understanding of the Seiberg-Witten equations must help to promote both the applications to physics and further generalizations.

In this paper, we shall generalize Campolattaro's viewpoint and set up the correspondence between the spinor and the curvature on the line bundle over the Spin_4^c manifold. We also set up the correspondence between the spinor and the 2–differential form on the Spin_4^c manifold. Seiberg-Witten equations and the perturbed Seiberg-Witten equations can then be well understood from this "direct" physical point of view.

We further suggest that the second Seiberg-Witten equation (18) (also the second perturbed Seiberg-Witten equation) can be considered as a nonlinear Dirac-like equation.

Since the connection is more basic than the curvature, we shall derive the spinor representation of the connection on the associated unitary line bundle on the Spin_4^c manifold.

2 SPINOR REPRESENTATION OF THE CURVATURE ON THE LINE BUNDLE OVER $Spin_4^c$ MANIFOLD

2.1 THE CASE OF CURVED SPACE–TIME

In order to obtain the spinor representation of the curvature of a unitary connection on the line bundle over Spin_4^c manifold, we first consider the case when the space–time X is a manifold with a pseudo-Riemannian metric of Lorentz signature. As described in more detail in (Misner, Thorne and Wheeler,1973), the Maxwell equations in the curved space–time are

$$\begin{aligned} F_{\mu\nu,\gamma} + F_{\nu\gamma,\mu} + F_{\gamma\mu,\nu} &= 0 \quad (19) \\ F^{\mu\nu}{}_{;\nu} &= j^{\mu} \quad (20) \end{aligned}$$

Where $\mu, \nu, \gamma = 0, 1, 2, 3$. These equations have the same form as in the case of Minkowski space–time, but with a comma replaced by a semicolon in Eq.(20). Here the semicolon stands for the covariant derivative.

One can naturally assume that there exists a spinor Ψ on the curved space–time X, such that

$$\begin{aligned} F^{\mu\nu} &= \overline{\Psi} S^{\mu\nu} \Psi \quad (21) \\ *F^{\mu\nu} &= \overline{\Psi} \gamma^5 S^{\mu\nu} \Psi \quad (22) \end{aligned}$$

where $\mu, \nu = 0, 1, 2, 3$. Notice that at this time Ψ and γ^μ is defined on the curved space-time.

We can now ask the question: When is iF the curvature of a unitary connction on some line bundle over the curved space-time X with the Hermitian metric?

One necessary condition is that F satisfy the Bianchi identity $dF = 0$, This is just one of the Maxwell equations (19). The other is the first Chern class $c_1(L)$ of a line bundle L over X is "quantized" – $c_1(L)$ integrates to an integer over any two dimensional cycle in X. This fact can be interpreted as requiring quantization of electric charge. Maintaining a perfect duality between electric and magnetic fields would then require quantization of magnetic charge as well.

From now on, we suppose that both electric and magnetic charges are quantized. The second condition is then automatically satisfied. We claim that these two conditions are also sufficient.

Now Eq.(21) are automatically the spinor representation of the curvature on the line bundle L over the curved space–time X.

2.2 THE CASE OF $Spin_4^c$ MANIFOLD

Let X be an oriented, closed four-dimensional Riemannian manifold. A Spin^c structure exists on any oriented four-manifold (Hirzebruch and Hopf, 1958; Lawson and Michelson, 1989).

The curvature F of a unitary connection on the line bundle L over Spin$_4^c$-manifold X satisfies the Bianchi identity,

$$dF = 0 \tag{23}$$

We also denote

$$\nabla_j F_{ij} = j_i \tag{24}$$

Here the covariant derivative $\nabla_j = \partial_j + \Gamma_j$, Γ_j is the Levi-Civita connection on X.

From the previous discussions, we have naturally the spinor representation of the curvature F,

$$\begin{aligned} F_{ij} &= W^\dagger S_{ij} W \\ &= \langle W, S_{ij} W \rangle \qquad (25) \\ *F_{ij} &= \langle W, \gamma_5 S_{ij} W \rangle \qquad (26) \end{aligned}$$

Here W is a spinor on X, $i, j = 1, 2, 3, 4$.

3 SPINOR REPRESENTATION OF 2–FORMS ON THE $Spin_4^c$ MANIFOLD

3.1 THE CASE OF MINKOWSKI SPACE–TIME

Campolattaro (1990a, 1990b) assumed that together with an electric current j_μ, there exists also a magnetic monopole current g_μ. Maxwell equations read

$$\begin{aligned} (*F'^{\mu\nu})_{,\nu} &= g^\mu \qquad (27) \\ F'^{\mu\nu}{}_{,\nu} &= j^\mu \qquad (28) \end{aligned}$$

There exists a spinor, such that

$$\begin{aligned} F'^{\mu\nu} &= \overline{\Psi} S^{\mu\nu} \Psi \qquad (29) \\ *F'^{\mu\nu} &= \overline{\Psi} \gamma^5 S^{\mu\nu} \Psi \qquad (30) \end{aligned}$$

It was shown that the spinor equation Eq.(13) in the presence of magnetic monopoles, reads

$$\gamma^\mu \Psi_{,\mu} = -i\gamma^\mu \frac{e^{\gamma^5 \alpha}}{\rho} \{Im(\overline{\Psi}_{,\mu}\Psi) - j_\mu - \gamma^5 [Im(\overline{\Psi}_{,\mu}\gamma^5\Psi) - g_\mu]\}\Psi \tag{31}$$

3.2 THE CASE OF CURVED SPACE–TIME

Given X, a four-dimensional manifold with a pseudo-Rimaniann metric of Lorentz signature. The generalized Maxwell equations read

$$\begin{aligned} (*F'^{\mu\nu})_{;\nu} &= g^{\mu} & (32) \\ F'^{\mu\nu}_{\ \ ;\nu} &= j^{\mu} & (33) \end{aligned}$$

One has the same expressions as in the previous discussions,

$$\begin{aligned} F'^{\mu\nu} &= \overline{\Psi} S^{\mu\nu} \Psi & (34) \\ *F'^{\mu\nu} &= \overline{\Psi} \gamma^5 S^{\mu\nu} \Psi & (35) \end{aligned}$$

Notice that at this time γ^{μ} and Ψ are defined in the curved space–time.

From Eq.(32), we state that the Bianchi identity is no longer satisfied: F' is only a 2–differential form on X. Given a 2-form F' one has

$$F' = F + \omega \qquad (36)$$

Here F is the curvature of a unitary connection on some line bundle over X, ω is a 2-form over X. Later we shall adopt this notation.

3.3 THE CASE OF $Spin^c_4$ MANIFOLD

Denote X a Spin^c_4 manifold. Given a 2-form $F + \omega$ on X. Denote

$$\begin{aligned} \nabla_j (*F_{ij} + *\omega_{ij}) &= g_i & (37) \\ \nabla_j (F_{ij} + \omega_{ij}) &= j_i & (38) \end{aligned}$$

We have natually the spinor representation

$$\begin{aligned} F_{ij} + \omega_{ij} &= \langle W, S_{ij} W \rangle & (39) \\ *F_{ij} + *\omega_{ij} &= \langle W, \gamma_5 S_{ij} W \rangle & (40) \end{aligned}$$

Here W is a spinor on X, $i, j = 1, 2, 3, 4$.

4 NONLINEAR DIRAC-LIKE EQUATION AND THE SECOND SEIBERG-WITTEN EQUATION

From Eqs.(25) and (26), The self-dual part of the curvature on L over the Spin^c_4 manifold X is

$$\begin{aligned} F^+_{ij} &= \frac{1}{2}(F_{ij} + *F_{ij}) \\ &= \langle W, \frac{1+\gamma_5}{2} S_{ij} W \rangle & (41) \end{aligned}$$

Denote $\frac{1+\gamma_5}{2} W = M$, Then

$$F^+_{ij} = \langle M, S_{ij} M \rangle \qquad (42)$$

Eq.(42) is just the second Seiberg-Witten equation (18). From Eqs.(23) and (24), one has

$$\nabla_i F^+_{ji} = \nabla_i \langle M, S_{ji} M\rangle = \frac{1}{2} j_j \tag{43}$$

Notice that $\widetilde{\nabla}_i S_{ji} = \nabla_i S_{ji}$. Eq.(42) has the equivalent form

$$Im\langle M, \gamma_j \widetilde{D} M\rangle = \langle M, \nabla_i S_{ji} M\rangle - Im\langle M, \widetilde{\nabla}_j M\rangle - \frac{1}{2} j_j \tag{44}$$

Just as Campolattaro's considerations, given F^+_{ij} thus j_j, one can verify that the positive chirality spinor M satisfy the nonlinear Dirac-like equation on the Spin^c_4 manifold X

$$\widetilde{D} M = -\frac{i}{\langle M, M\rangle}\{\langle M, \nabla_k S_{ik} M\rangle - Im\langle M, \widetilde{\nabla}_i M\rangle - \frac{1}{2} j_i\}\gamma_i M \tag{45}$$

Since the term $\langle M, \gamma_j \gamma_i M\rangle$ is pure imaginary, if $i \neq j$. One can show that Eq.(45) is the sufficient condition of Eq.(44), thus is the sufficient condition of the second Seiberg-Witten equation (42).

Notice that Seiberg-Witten equations

$$\begin{aligned} \widetilde{D} M &= 0 \\ F^+_{ij} &= \langle M, S_{ij} M\rangle \end{aligned}$$

are equivalent to

$$\begin{aligned} \widetilde{D} M &= 0 \\ \widetilde{D} M &= -\frac{i}{\langle M, M\rangle}\{\langle M, \nabla_k S_{ik} M\rangle - Im\langle M, \widetilde{\nabla}_i M\rangle - \frac{1}{2} j_i\}\gamma_i M \end{aligned}$$

From this point of view, the second Seiberg-Witten equation can be replaced by the nonlinear Dirac-like equation (45).

5 NONLINEAR DIRAC-LIKE EQUATION AND THE PERTURBED SECOND SEIBERG-WITTEN EQUATION

From Eqs.(39) and (40), write $\phi = \omega^+ = \frac{1}{2}(\omega + *\omega)$, one has

$$F^+_{ij} + \phi_{ij} = \langle M, S_{ij} M\rangle \tag{46}$$

Eq.(46) is just the second perturbed Seiberg-Witten equation. From Eqs.(37) and (38), we have

$$\nabla_i \langle M, S_{ji} M\rangle = \frac{j_j + g_j}{2} \tag{47}$$

Just as the previous discussions, given $j_j + g_j$, we have a nonlinear Dirac-like equation on the Spin^c_4 manifold X,

$$\widetilde{D} M = -\frac{i}{\langle M, M\rangle}\{\langle M, \nabla_k S_{ik} M\rangle - Im\langle M, \widetilde{\nabla}_i M\rangle - \frac{j_i + g_i}{2}\}\gamma_i M \tag{48}$$

This equation is the sufficient condition of the second perturbed Seiberg-Witten equation (46). Notice that the perturbed Seiberg-Witten equations

$$\begin{aligned} \widetilde{D}M &= 0 \\ F_{ij}^{+} + \phi_{ij} &= \langle M, S_{ij}M \rangle \end{aligned}$$

are equivalent to

$$\begin{aligned} \widetilde{D}M &= 0 \\ \widetilde{D}M &= -\frac{i}{\langle M, M \rangle}\{\langle M, \nabla_k S_{ik} M \rangle - Im\langle M, \widetilde{\nabla}_i M \rangle - \frac{j_i + g_i}{2}\}\gamma_i M \end{aligned}$$

So one can say that it is important to study the nonlinear Dirac-like equations (45) and (48).

6 SPINOR REPRESENTATION OF $U(1)$ CONNECTION ON $Spin_4^c$ MANIFOLD

Now we derive the spinor representation of the connection A on the line bundle L over Spin$_4^c$ manifold X. Choose a positive chirality spinor M, which is the solution of the Dirac equation

$$\gamma_i \widetilde{\nabla}_i M = 0 \tag{49}$$

One has

$$\langle M, \gamma_j \gamma_i \widetilde{\nabla}_i M \rangle = 0 \tag{50}$$

which is equal to

$$\langle M, S_{ji} \widetilde{\nabla}_i M \rangle - \frac{i}{2}\langle M, \widetilde{\nabla}_j M \rangle = 0 \tag{51}$$

By taking the Hermitian conjugation of Eq.(51), one has

$$\langle \widetilde{\nabla}_i M, S_{ji} M \rangle + \frac{i}{2}\langle \widetilde{\nabla}_j M, M \rangle = 0 \tag{52}$$

Notice that $\widetilde{\nabla}_i S_{ji} = \nabla_i S_{ji}$. By adding Eqs.(51) and (52), one obtains,

$$\nabla_i \langle M, S_{ji} M \rangle = \langle M, (\nabla_i S_{ji}) M \rangle - Im\langle M, \widetilde{\nabla}_j M \rangle \tag{53}$$

Eq.(53) is completely equivalent to the Dirac Eq.(49).

Since the computation is local, we can write

$$M = \Psi \otimes \lambda \tag{54}$$

where $\Psi \in S_0^+$ is a local positive chirality spinor on X, and $\lambda \in L^{\frac{1}{2}}$. Notice that

$$\langle \Psi_1 \otimes \lambda_1, \Psi_2 \otimes \lambda_2 \rangle = \langle \Psi_1, \Psi_2 \rangle \langle \lambda_1, \lambda_2 \rangle \tag{55}$$

Eq.(53) reads

$$\begin{aligned}\nabla_i[\langle\Psi, S_{ji}\Psi\rangle\langle\lambda,\lambda\rangle] &= \langle\Psi,(\nabla_i S_{ji})\Psi\rangle\langle\lambda,\lambda\rangle \\ &\quad - Im\langle\Psi\otimes\lambda, \nabla_j\Psi\otimes\lambda + \Psi\otimes\nabla_j^{'}\lambda\rangle\end{aligned} \tag{56}$$

Here $\nabla_j = \partial_j + \Gamma_j$. $\nabla_j^{'} = \partial_j + ia_j$, a_j is the connection on $L^{\frac{1}{2}}$. We have

$$\begin{aligned}a_j &= -\frac{Im\langle\Psi,\nabla_j\Psi\rangle}{\langle\Psi,\Psi\rangle} + \frac{\langle\Psi,(\nabla_i S_{ji})\Psi\rangle}{\langle\Psi,\Psi\rangle} \\ &\quad - \frac{\nabla_i[\langle\Psi,S_{ji}\Psi\rangle\langle\lambda,\lambda\rangle]}{\langle\Psi,\Psi\rangle\langle\lambda,\lambda\rangle} - \frac{Im\langle\lambda,\partial_j\lambda\rangle}{\langle\lambda,\lambda\rangle}\end{aligned} \tag{57}$$

Since L is unitary, i.e., $\langle\lambda,\lambda\rangle = 1$. One has

$$Im\langle\lambda,\partial_j\lambda\rangle = -i\lambda^{-1}\partial_j\lambda$$

The transformation rule of a_j is

$$ia_j' = ia_j + \lambda^{-1}\partial_j\lambda \tag{58}$$

We also denote a' by a. It leads to

$$\begin{aligned}a_j &= -\frac{Im\langle\Psi,\nabla_j\Psi\rangle}{\langle\Psi,\Psi\rangle} + \frac{\langle\Psi,(\nabla_i S_{ji})\Psi\rangle}{\langle\Psi,\Psi\rangle} - \frac{\nabla_i\langle\Psi,S_{ji}\Psi\rangle}{\langle\Psi,\Psi\rangle} \\ &= \frac{Im\langle\Psi,\gamma_j D\Psi\rangle}{\langle\Psi,\Psi\rangle}.\end{aligned} \tag{59}$$

Here $D = \gamma_i\nabla_i$ is the Dirac operator acting on the local positive chirality spinor Ψ. This means that the Dirac equation (49) can be locally reduced to

$$\gamma_j(\nabla_j + ia_j)\Psi = 0 \tag{60}$$

Notice that $da = \frac{1}{2}F$. For any unitary connection A on L, $F = dA$. we can choose

$$A = 2a \tag{61}$$

Finally, one has the spinor representation of A,

$$A_j = 2\frac{Im\langle\Psi,\gamma_j D\Psi\rangle}{\langle\Psi,\Psi\rangle}. \tag{62}$$

Where $j = 1, 2, 3, 4$.

7 CONCLUSIONS

We have analysed the relation between the spinor and the curvature on a unitary line bundle over the Spin_4^c manifold. The relation between the spinor and the 2–form on the Spin_4^c manifold has also been considered.

We have proposed two nonlinear Dirac-like equations which are equivalent to the second Seiberg-Witten equation and the second perturbed Seiberg-Witten equation respectively. This means one can study (the self dual part of) the curvature (or the 2-form) in the quantum mechanics level.

We have also derived the local spinor representation of the unitary connection on the associated line bundle over the Spin_4^c manifold.

These discussions can be generalized to the non-Abelian cases.

The relation between the properties of the nonlinear Dirac-like equation and the moduli space of the (perturbed) Seiberg-Witten equations need to be further studied.

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